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# Autohomeomorphisms of compact groups

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*Abstract*

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Every infinite compact group is shown to have autohomeomorphisms [1] which do not preserve its Haar measure.

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In [2] van Douwen constructed an infinite, compact, totally disconnected, homogeneous Hausdorff space $X$ which carries a positive Borel measure $\mu$ such that $\mu(h(E))=\mu(E)$ for every Borel set $E\subset X$ and every autohomeomorphism $h$ of $X$. (In other words, $\mu$ is *preserved* by every $h$.) Moreover, two compact open subsets of $X$ are homeomorphic if and only if they have the same measure.

He then raised the question whether the same could be true with some compact group $G$ and its Haar measure in place of $X$ and $\mu$:

*Does there exist an infinite compact connected group $G$ such that every autohomeomorphism of $G$ preserves its Haar measure?*

He asked for a connected $G$ because he could show that the answer is negative in the totally disconnected case. However, it is always negative:

**Theorem.** *If $G$ is an infinite compact group, with Haar measure $m_G$, then there is an autohomeomorphism $h$ of $G$ such that*

$$m_G(h(E)) \neq m_G(E)$$

*for some open set $E\subset G$.*

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[1] This term seems to have been originated by van Douwen.

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**Proof.** We shall exploit the fact that the conclusion of the theorem is obviously true when $G$ is locally Euclidean.

Let $G_0$ be the component of $G$ which contains the identity element $e$. Since the totally disconnected case was settled by van Douwen we assume, without loss of generality, that $G_0 \neq \{e\}$.

The theorem stated on p. 99 of [1] asserts that every neighborhood $U$ of $e$ contains a compact normal subgroup $H$ of $G$ with the property that $G/H$ is isomorphic to a group of linear operators on some $R^n$. Thus $G/H$ is a Lie group [1, p. 82], hence is locally Euclidean.

Since $G_0 \neq \{e\}$ we can take $U$ so small that $U$ does not contain $G_0$. Then if $\varphi$ is the canonical homeomorphism from $G$ onto $G/H$, the connected set $\varphi(G_0)$ is not a singleton. The dimension $N$ of $G/H$ is therefore positive. It follows (see pp. 193–194 of [1]) that there is an $N$-cell $W$ in $G$ such that the restriction of $\varphi$ to $W$ is a homeomorphism onto an open neighborhood $V$ of the identity element in $G/H$. Let $\psi : V \to W$ be the inverse of this homeomorphism.

If $x \in \varphi^{-1}(V)$ and $z = \psi(\varphi(x))$ then $z \in W$ and $\varphi(z) = \varphi(x)$. Thus $xz^{-1} \in H$, and we conclude that

$$x = yz, \quad y \in H, \; z \in W. \tag{$*$}$$

Moreover, the factorization $(*)$ is unique since $\varphi$ is one-to-one on $W$. (Thus $\varphi^{-1}(V)$ is homeomorphic to $H \times W$.)

To finish, let $f$ be an autohomeomorphism of the locally Euclidean group $G/H$ which fixes every point outside some compact subset of $V$ but does not preserve the Haar measure $m_{G/H}$ within $V$. Define $h : G \to G$ as follows:

(a) $h(x) = x$ if $x \in G \setminus \varphi^{-1}(V)$.

(b) If $x \in \varphi^{-1}(V)$ and $x = yz$ as in $(*)$, put

$$h(x) = y \cdot [\psi(f(\varphi(z)))].$$

Since $\psi \circ f \circ \varphi$ is a homeomorphism of $W$ onto $W$, $h$ is a homeomorphism of $G$ onto $G$.

Since, in $\varphi^{-1}(V)$, $m_G = m_H \times m_{G/H}$, our choice of $f$ shows that $h$ does not preserve $m_G$ within $\varphi^{-1}(V)$. □

## References

[1] D. Montgomery and L. Zippin, Topological Transformation Groups (Interscience, New York, 1955).

[2] E.K. van Douwen, A compact space with a measure that knows which sets are homeomorphic, Adv. in Math. 52 (1984) 1–33.